Applications of Laplace-Beltrami operator for Jack polynomials

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Abstract. We use a new method to study the Laplace-Beltrami type operator on the Fock space of symmetric functions, and as an example of our explicit computation we show that the Jack symmetric functions are the only family of eigenvectors of the differential operator. As applications of this explicit method we find a combinatorial formula for Jack symmetric functions and the Littlewood-Richardson coefficients in the Jack case. As further applications, we obtain a new determinantal formula for Jack symmetric functions. We also obtained a generalized raising operator formula for Jack symmetric functions, and a formula for the explicit action of Virasoro operators. Special cases of our formulas imply Mimachi-Yamada's result on Jack symmetric functions of rectangular shapes, as well as the explicit formula for Jack functions of two rows or two columns.

1. Introduction

For $\alpha \in \mathbb{C}$ the generalized Laplace-Beltrami operator

$$L(\alpha) = \frac{\alpha}{2}\sum_{i=1}^{n}(x_i\frac{\partial}{\partial x_i})^2 + \frac{1}{2}\sum_{i<j}^{n}\frac{x_i+x_j}{x_i-x_j}(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}) \tag{1.1}$$

is the Hamiltonian for the Calogero-Sutherland-Moses model. Macdonald showed that the eigenstates are the Jack symmetric polynomials [**Ja**] in variables $x_1, x_2, \cdots, x_n$. When $\alpha = 1, 2, 1/2$ the Laplace-Beltrami operator is the radial Laplace operator for the symmetric spaces over the complex, real and quaternion fields. Like Schur functions, the Jack functions are also polynomials in power sum symmetric functions $p_k = \sum_{i=1}^n x_i^k$, $k = 1, \cdots, n$. Moreover it is a fundamental fact that these symmetric functions enjoy the stability property that $P_\lambda(x_1, \cdots, x_n; \alpha)$ is the same polynomial as long as

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$n > |\lambda|$, i.e., they are actually polynomials in variables $p_k = \sum_{i=1}^{\infty} x_i^k$. It is advantageous to view $P_\lambda(x;\alpha)$ as an element in $\mathbb{Q}(\alpha)[p_1, p_2, \cdots]$.

It was shown by Soko [**So**] that

$$L(\alpha)(m_\lambda) = \left[\sum_{i=1}^{l(\lambda)} \left(\frac{\alpha}{2}\lambda_i^2 + \frac{1}{2}(n+1-2i)\lambda_i\right)\right] m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu. \tag{1.2}$$

The action of the finite Laplace operator can be used to study Jack symmetric functions, for instance a determinant formula [**LLM**] is a direct consequence. It is clear that a direct generalization of this finite Laplace operator will not work as the eigenvalue does not make sense when $n \longrightarrow \infty$.

In this paper we consider a Laplace-Beltrami like operator in infinitely many variables which has all the favorite properties enjoyed by the finite Laplace-Beltrami operator, moreover the eigenvalues of the new operator can be used to distinguish eigenstates. Our starting point is the observation that the space $\Lambda = \mathbb{C}[p_1, p_2, p_3 \ldots]$ is actually the Fock space for the infinite dimensional Heisenberg algebra generated by h_n with relations

$$[h_m, h_n] = m\alpha\delta_{m,-n} Id.$$

If we identify h_{-n} with p_n, the ring of symmetric functions is isomorphic to the Fock space of the Heisenberg Lie algebra, which is the canonical irreducible representation of the latter. Under this identification the generating function of the generalized homogeneous symmetric functions q_n is exactly half of the vertex operator [**J2, CJ**].

There is an indirect method to show that the operator diagonalize Jack functions. For our later purpose we still provide a direct method to show that $P_\lambda(\alpha;x)$ are eigenfunctions of the following graded differential operator

$$\sum_{i,j\geq 1} ij\alpha^2 p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} + \sum_{i,j\geq 1} (i+j)\alpha p_i p_j \frac{\partial}{\partial p_{i+j}} + \alpha(\alpha-1)\sum_{i\geq 1} i^2 p_i \frac{\partial}{\partial p_i}$$

and the eigenvalues can distinguish Jack functions.

Using our new method, we derive an explicit action of $D(\alpha)$ on the basis of generalized homogeneous functions $q_\lambda(\alpha)$ or monomial symmetric functions m_λ, which establish a priori the triangularity of the transition matrix between these bases and Jack polynomials in one step. We then use the differential operator to give

(i) an explicit iteration formula for the coefficients of Q_λ in terms of q_λ;

(ii) a combinatorial formula for Littlewood-Richardson coefficient and provide a formula for Stanley's conjecture for Jack polynomials;

(iii) reformulation of Stanley formula for two columns and Jing-Józefiak formula for two rows.

It is well-known that Jack polynomials span a representation of Virasoro algebra and the extremal vectors or singular vectors are the Jack polynomials of rectangular shapes. This result was originally proved by Mimachi-Yamada using differential equations and it was done over finitely many variables.

Adopting the same idea as in the Laplace-Beltrami like operator, we give an explicit action of Virasoro algebra on Jack functions which confirms several conjectural formulas made by Sakamoto et al [**SAFR**]. As a consequence we obtain a simpler proof of Mimachi-Yamada's result using the Feigin-Fuchs realization.

This paper is organized as follows. In section 2, we recall some basic notions about symmetric functions. We introduce the differential operator of Laplace-Beltrami type in section 3 and compute its action on generalized homogeneous polynomials to give a new characterization of Jack symmetric functions. In section 4 we derive a raising operator formula for the action of Laplace-Beltrami operator. Next in section 5 we derive several applications of our new differential operator: a determinant formula for Jack symmetric functions, an iterative formula for the transition matrix from generalized homogeneous functions or monomial symmetric functions to Jack functions, and explicit action of Virasoro operators on Jack functions. We also show how our method can be used to give combinatorial formulas for Jack symmetric functions and then for generalized Littlewood-Richardson coefficients.

2. Jack functions

We first recall a few basic notions about symmetric functions [**M, S**]. A partition λ is a sequence of nonnegative integers, written usually in decreasing order as $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s)$, sometimes also as $(1^{m_1}\ 2^{m_2} \cdots)$, where $m_i = m_i(\lambda)$ is the multiplicity of i occurring in the parts of λ. The length of λ, denoted as $l(\lambda)$, is the number of non-zero parts in λ, and the weight $|\lambda|$ is $\sum_i im_i$. It is convenient to denote $m(\lambda)! = m_1!m_2!\cdots$, $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$. A partition λ of weight n is usually denoted by $\lambda \vdash n$. The set of all partitions is denoted as $\mathcal{P}$. For two partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\mu = (\mu_1, \mu_2, \cdots)$ of the same weight, we say that $\lambda \geq \mu$ if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i, this defines what's called dominance ordering. There is a canonical total ordering on $\mathcal{P}_n$, the reverse lexicographic ordering. For $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\mu = (\mu_1, \mu_2, \cdots)$ in $\mathcal{P}_n$, we say that λ is greater than μ in reverse lexicographic ordering, denoted as $\lambda >^L \mu$, if the first non-vanishing difference $\lambda_i - \mu_i$ is positive, and $\lambda \geq^L \mu$ means $\lambda >^L \mu$ or $\lambda = \mu$. Sometimes λ is identified with its Young diagram $\lambda = \{(i,j) | 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$. Thus $\mu \subseteq \lambda$ if and only if $\lambda_i \geq \mu_i$ for all i, and in this case we define skew partition λ/μ as the set difference of λ and μ. We say that λ/μ is a horizontal-n strip if it contains n squares with no two squares in the same column. The conjugate of λ, $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$, is a partition whose diagram is the transpose of the diagram of λ, hence λ'_i is the number of the j's such that $\lambda_j \geq i$. For square $s = (i,j) \in \lambda$, the lower hook-length $h_*^\lambda(s)$ is defined to be $\alpha(\lambda_i - j) + (\lambda'_j - i + 1)$, and the upper hook-length $h^*_\lambda(s) = \alpha(\lambda_i - j + 1) + (\lambda'_j - i)$.

Definition 2.1. For a subset S of partition λ, define $h_*^\lambda(S)$ as $\prod_{s\in S} h_*^\lambda(s)$, and similarly for $h^*_\lambda(S)$. For partitions $\mu \subseteq \lambda$, we say that a square s is

bottomed if s is in the column which contains at least one square of λ/μ, and it's un-bottomed otherwise. We denote μ_b (resp. λ_b) as the set of the bottomed squares of μ(resp. λ), and μ_u (resp. λ_μ) as the set of the un-bottomed squares of μ (resp. λ).

The ring Λ of symmetric functions is a $\mathbb{Z}$-module with basis $m_\lambda = \sum x_{i_1}^{\lambda_1}\cdots x_{i_k}^{\lambda_k}$, $\lambda \in \mathcal{P}$. The power sum symmetric functions $p_\lambda = p_{\lambda_1}\cdots p_{\lambda_k}$ form a basis of $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $F = \mathbb{Q}(\alpha)$ be the field of rational functions in indeterminate α. The Jack polynomial [**Ja**] is a special orthogonal symmetric function of $\Lambda_F = \Lambda \otimes_{\mathbb{Z}} F$ under the following inner product. For two partitions $\lambda, \mu \in \mathcal{P}$ the (Jack) scalar product on Λ_F is given by

$$\langle p_\lambda, p_\mu\rangle = \delta_{\lambda,\mu}\alpha^{l(\lambda)} z_\lambda, \tag{2.1}$$

where δ is the Kronecker symbol.

The Jack symmetric functions $P_\lambda(\alpha)$ for $\lambda \in \mathcal{P}$ are defined by the following [**M**]:

$$P_\lambda(\alpha) = \sum_{\lambda \geq \mu} c_{\lambda\mu}(\alpha) m_\mu,$$
$$\langle P_\lambda(\alpha), P_\mu(\alpha\rangle = 0 \text{ for } \lambda \neq \mu,$$

where $c_{\lambda\mu}(\alpha) \in F$ $(\lambda, \mu \in \mathcal{P})$ and $c_{\lambda\lambda}(\alpha) = 1$.

Defined by $\langle Q_\lambda(\alpha), P_\mu(\alpha)\rangle = \delta_{\lambda,\mu}$, the dual Jack function $Q_\lambda(\alpha) = \langle P_\lambda, P_\lambda\rangle^{-1} P_\lambda(\alpha)$. Another normalization $J_\lambda(\alpha)$ of Jack symmetric function is also useful. Let

$$J_\lambda(\alpha) = \sum_{\nu \leq \lambda} v_{\lambda\nu}(\alpha) m_\nu,$$

with the normalization defined by $v_{\lambda,(1^{|\lambda|})} = |\lambda|!$.

The generalized homogeneous symmetric functions of Λ_F are defined by

$$q_\lambda(\alpha) = Q_{\lambda_1}(\alpha) Q_{\lambda_2}(\alpha)\cdots Q_{\lambda_l}(\alpha), \tag{2.2}$$

where $Q_{(n)}(\alpha)$, simplified as $Q_n(\alpha)$, is known as

$$Q_n(\alpha) = \sum_{\lambda \vdash n} \alpha^{-l(\lambda)} z_\lambda^{-1} p_\lambda. \tag{2.3}$$

Thus $Q_i(\alpha) = 0$ for $i < 0$ and $Q_0(\alpha) = 1$. Its generating function is:

$$Y(z) = exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n\alpha} p_n\Big) = \sum_n Q_n(\alpha) z^n.$$

For convenience we may omit the parameter α in $Q_n(\alpha)$ and $q_\lambda(\alpha)$, simply write them as Q_n and q_λ. The following theorem of Stanley will be needed in our paper.

Theorem 2.2. [**S**] *For partitions μ ,λ, and positive integer n, $\langle J_n J_\mu, J_\lambda\rangle \neq 0$ if and only if $\mu \subseteq \lambda$ and λ/μ is a horizontal n-strip. And in this case we have*

$$\langle J_n(\alpha)J_\mu(\alpha), J_\lambda(\alpha)\rangle = h_*^\mu(\mu_u)h_\mu^*(\mu_b)\cdot h_n^*(n)\cdot h_\lambda^*(\lambda_u)h_*^\lambda(\lambda_b). \tag{2.4}$$

We also list some useful properties of q_λ and J_λ as follows.

Lemma 2.3. [**S**] *For any partition λ, ν, one has*

$$\begin{aligned}
&\langle q_\lambda(\alpha), m_\nu\rangle = \delta_{\lambda\nu},\\
&q_\lambda(\alpha) = Q_\lambda(\alpha) + \sum_{\mu>\lambda} c'_{\lambda\mu}Q_\mu(\alpha),\\
&J_\lambda(\alpha) = h_\lambda^*(\lambda)Q_\lambda(\alpha),\\
&J_\lambda(\alpha) = h_*^\lambda(\lambda)P_\lambda(\alpha),
\end{aligned}$$

where $c'_{\lambda\mu} \in F$.

Knop and Sahi proved that (see also [**HHL**]).

Theorem 2.4. [**KS**] *Let $J_\lambda(\alpha) = \sum_{\mu\leq\lambda} v_{\lambda\nu}(\alpha)m_\mu$, then we have $\frac{v_{\lambda\mu}(\alpha)}{m(\mu)!} \in \mathbb{Z}_{\geq 0}[\alpha]$.*

In [**EJ**], some transition matrices were given using combinatorial methods, the following is a special case of such matrices.

Proposition 2.5. [**EJ**] *For $\mu \vdash n$, set $m_\mu = \sum_{\lambda\geq\mu} T_{\mu\lambda}p_\lambda$, then we have $T_{\mu\lambda}m(\mu)! \in \mathbb{Z}$ and $T_{\mu,(n)} = (-1)^{l(\mu)-1}\frac{(l(\mu)-1)!}{m(\mu)!}$.*

Combining Theorem 2.4 with Proposition 2.5, it is easy to see one of Stanley's conjectures:

Theorem 2.6. [**S**] *Set $J_\lambda(\alpha) = \sum_\mu c_{\lambda\mu}(\alpha)p_\mu$, then we have*

$$c_{\lambda\mu}(\alpha) \in \mathbb{Z}[\alpha].$$

A direct consequence of this is

Corollary 2.7.

$$C^\lambda_{\mu\nu}(\alpha) = \langle J_\mu(\alpha)J_\nu(\alpha), J_\lambda(\alpha)\rangle \in \mathbb{Z}[\alpha].$$

3. Laplace-Beltrami type operator for Jack functions

Λ_F is a graded ring with gradation given by the degree, and let $\Lambda_F(m) = \{f \in \Lambda_F | deg(f) = m\}$, thus $\Lambda_F = \oplus_{n=0}^\infty \Lambda_F(n)$. A linear operator A is called a graded operator of degree n if $A.\Lambda_F(m) \subset \Lambda_F(m+n)$.

The Heisenberg algebra H_α is an infinite dimensional Lie algebra generated by h_n ($n \neq 0$), satisfying the relations:

$$[h_m, h_n] = m\alpha\delta_{m,-n}.$$

For $n>0$, identifying h_{-n} with p_n in Λ_F, we have the canonical representation of H_α on Λ_F defined by:

$$h_n.v = n\alpha\frac{\partial}{\partial h_{-n}}(v),$$
$$h_{-n}.v = h_{-n}v,$$

for $n>0$ and $v\in\Lambda$.

The operator h_n is then a graded operator of degree $-n$.

On the ring Λ_F of symmetric functions, we introduce the following graded operator of degree 0

$$D(\alpha)=\sum_{i,j\geq 1}(i+j)\alpha p_i p_j\frac{\partial}{\partial p_{i+j}}+\sum_{i,j\geq 1} ij\alpha^2 p_{i+j}\frac{\partial^2}{\partial p_i\partial p_j}+\alpha(\alpha-1)\sum_{k\geq 1}k^2 p_k\frac{\partial}{\partial p_k},$$

where the infinite sum is well-defined on each subspace $\Lambda(m)$.

Using Heisenberg relations the operator $D(\alpha)$ can be rewritten as

$$D(\alpha)=\sum_{i,j\geq 1}h_{-i}h_{-j}h_{i+j}+\sum_{i,j\geq 1}h_{-(i+j)}h_i h_j+(\alpha-1)\sum_{i\geq 1} ih_{-i}h_i.$$

Remark 3.1. A similar operator was used in physics literature (eg. [**AMOS**], [**I**]) to study Virasoro constraints. When $\alpha=1$, the operator $D(1)$ was also studied by [**FW**] for the Schur case. We will see that the third term is crucial in the Jack case.

We now give a characterization of Jack functions using the Laplace-Beltrami operator.

Theorem 3.2. *For $\lambda=(\lambda_1,\lambda_2,\cdots)\in\mathcal{P}$, $Q'_\lambda(\alpha)=Q_\lambda(\alpha)$ if and only if the following properties are satisfied:*

(1) There are rational functions $C_{\lambda\mu}(\alpha)$ of α with $C_{\lambda\lambda}(\alpha)=1$ such that

$$Q'_\lambda(\alpha)=\sum_{\mu\geq\lambda}C_{\lambda\mu}(\alpha)q_\mu(\alpha),$$

(2) $Q'_\lambda(\alpha)$ is an eigenvector for $D(\alpha)$.

Moreover, $D(\alpha).Q_\lambda(\alpha)=e_\lambda(\alpha)Q_\lambda(\alpha)$, where

$$e_\lambda(\alpha)=\alpha^2\sum_i\lambda_i^2+\alpha(|\lambda|-2\sum_i i\lambda_i).$$

Definition 3.3. For an operator S on Λ_F, define the conjugate of S, denoted as S^*, by $\langle S.u,v\rangle=\langle u,S^*.v\rangle$ for all $u,v\in\Lambda_F$. S is called self-adjoint, if $S=S^*$. Let S be a degree 0 graded operator on Λ_F, we say that S is a raising operator on q_λ's if $S.q_\lambda=\sum_{\mu\geq\lambda}C_{\lambda\mu}q_\mu$ for every $\lambda\in\mathcal{P}$.

It's easy to prove the following Lemma, mainly using Lemma 2.3.

Lemma 3.4. *A graded operator S on Λ_F has Jack functions as its eigenvectors if and only if S is a raising operator on q_λ's and S is self-adjoint.*

To prove Theorem 3.2, we will show that $D(\alpha)$ is a raising operator on q_λ's as $D(\alpha)$ is obviously self-adjoint. First, let us look at the one row case.

Lemma 3.5. *The Jack functions $Q_n(\alpha)$'s are eigenvector of $D(\alpha)$, explicitly we have*

$$D(\alpha).Q_n(\alpha) = (\alpha^2 n^2 - n\alpha)Q_n(\alpha). \tag{3.1}$$

PROOF. Recall that $Q_n(\alpha) = \sum_{\lambda\vdash n} \alpha^{-l(\lambda)} z_\lambda^{-1} p_\lambda$. For $\mu = (1^{m_1}2^{m_2}\cdots)$, the coefficient of p_μ in $D(\alpha).Q_n(\alpha)$ is

$$\begin{aligned}
&(\alpha-1)\sum_{k\geq 1}\alpha k^2 m_k \alpha^{-l(\mu)} z_\mu^{-1}\\
&+\sum_{i,j\geq 1}\alpha(i+j)(m_{i+j}+1)\alpha^{-(l(\mu)-1)} z_\mu^{-1} i m_i j(m_j-\delta_{i,j})(i+j)^{-1}(m_{i+j}+1)^{-1}\\
&+\sum_{i,j\geq 1}\alpha i\alpha j(m_i+1)\alpha^{-(l(\mu)+1)} z_\mu^{-1} i^{-1} j^{-1}(m_j+1+\delta_{i,j})^{-1}(i+j)m_{i+j}\\
&=\alpha^{-l(\mu)} z_\mu^{-1}\Big[(\alpha-1)\alpha\sum_{k\geq 1}k^2 m_k+\sum_{i,j\geq 1}\alpha^2 i m_i j(m_j-\delta_{i,j})+\sum_{i,j\geq 1}\alpha(i+j)m_{i+j}\Big]\\
&=\alpha^{-l(\mu)} z_\mu^{-1}\Big(\alpha^2|\mu|^2-\alpha|\mu|\Big).
\end{aligned}$$

□

Note that the first and third summand of $D(\alpha)$ are derivations on Λ_F. The second term is a second order differential operator. The following lemma, Lemma 3.8, assists in computing its action on products. First we need the following corollary of Proposition 2.5.

Lemma 3.6. *Let $p_n = \sum_{\mu\vdash n} a_{n,\mu} q_\mu(\alpha)$, then we have*

$$a_{n,\mu} = n\alpha(-1)^{l(\mu)-1}(l(u)-1)!/m(\mu)!.$$

PROOF.

$$\begin{aligned}
a_{n,\mu} &= \langle h_{-n}.1, m_\mu\rangle = n\alpha\langle 1, h_n.m_\mu\rangle = \langle 1, \frac{\partial}{\partial h_{-n}} m_\mu\rangle = n\alpha T_{\mu,(n)}\\
&= n\alpha(-1)^{l(\mu)-1}(l(u)-1)!/m(\mu)!.
\end{aligned}$$

□

Applying h_m to two sides of $Y(z) = exp\Big(\sum_{n=1}^\infty \frac{z^n}{n\alpha} h_{-n}\Big) = \sum_n Q_n(\alpha) z^n$, we have the following lemma about the action of h_n on one-row Jack functions.

Lemma 3.7. *For positive integer m and integer n, we have $h_m.Q_n = Q_{n-m}$.*

Lemma 3.8. *For positive integers m and n with $m \geq n$, set*

$$A_{m,n} = \sum_{i,j\geq 1} h_{-(i+j)}(h_i.Q_m)(h_j.Q_n) = \sum_\lambda b_\lambda q_\lambda,$$

then $b_\lambda = [m'(1-\delta_{m,m'}) - n']\alpha$ *for* $\lambda = (m', n') \geq (m, n)$, *and* $b_\lambda = 0$ *otherwise.*

PROOF. By Lemma 3.7, we have

$$A_{m,n} = \sum_{i,j\geq 1} h_{-(i+j)} q_{m-i} q_{n-j}.$$

Let's consider the three cases of λ: $l(\lambda) = 1$, $l(\lambda) = 2$, and $l(\lambda) \geq 3$.

First, for $l(\lambda) = 1$, we have $\lambda = (m+n, 0) \geq (m, n)$. In the summation of $A_{m,n}$, only $(i, j) = (m, n)$ contributes to q_λ. By Lemma 3.6, the coefficient is $a_{(m+n),(m+n)} = (m + n)\alpha$. This is in accord with the statement of the lemma.

Second, for $l(\lambda) = 2$, set $\lambda = (m', n')$. If we don't have $\lambda \geq (m, n)$, then we have $m > m' > n' > n$ or $m > m' = n' > n$. If $m > m' > n' > n$, the coefficient of q_λ in $A_{m,n}$ is

$$a_{n',n'} + a_{m',m'} + a_{m'+n',(m',n')} = n'\alpha + m'\alpha + (m' + n')\alpha(-1) = 0.$$

It can be found similarly that the coefficient is also zero if $m > m' = n' > n$. If we do have $\lambda \geq (m, n)$, it can be found similarly that $(m' - n')\alpha$ if $m' > m \geq n > n'$, and the coefficient is $-n'\alpha$ if $m' = m \geq n = n'$.

Third, for $l(\lambda) \geq 3$, set $\lambda = (\lambda_1, \lambda_2, \cdots) = (1^{m_1} 2^{m_2} \cdots)$. The coefficient of q_λ is

$$\begin{aligned}(3.2)\qquad & \alpha(-1)^{l(\lambda)}(l(\lambda) - 3)!(m(\lambda)!)^{-1}\Big[-(m + n)(l(\lambda) - 1)(l(\lambda) - 2) \\ &+ \sum_{j=1}^{n-1}(m + n - j)(l(\lambda) - 2)m_j + \sum_{i=1}^{m-1}(m + n - i)(l(\lambda) - 2)m_i \\ &+ \sum_{i=1}^{m-1}\sum_{j=1}^{n-1}(m + n - i - j)(-1)m_i(m_j - \delta_{i,j})\Big].\end{aligned}$$

For convenience we denote the term inside of the square bracket as A. The four summands in A correspond to the four kinds of assignment of (i, j), with the first one corresponding to $(i, j) = (m, n)$, the second corresponding to $i = m$,$j = 1, \cdots, n - 1$, the third to $i = 1, \cdots, m - 1$, $j = n$, and the last to $i = 1, \cdots, m - 1$, $j = 1, \cdots, n - 1$. We need to prove that $A = 0$.

In the first subcase that $\lambda_1 \leq n - 1$, A is equal to

$$\begin{aligned} A_1 = &-(m + n)(l(\lambda) - 1)(l(\lambda) - 2) + 2(m + n)(l(\lambda) - 1)(l(\lambda) - 2) \\ &- (m + n)(l(\lambda) - 1)(l(\lambda) - 2) \\ = &\ 0,\end{aligned}$$

where we use the property that $\sum_i m_i = l(\lambda)$, and $\sum_i i m_i = m + n$.

In the second subcase that $\lambda_1 \geq m$, and the third subcase that $m > \lambda_1 \geq n$, A can be proved similarly to be zero. $\square$

4. Raising operator formula for Laplace-Beltrami operator

The differential operator $D(\alpha)$ acts triangularly on the generalized homogeneous polynomials.

Proposition 4.1. *The action of $D(\alpha)$ on q_λ is given explicitly as follows.*

$$\begin{aligned}(4.1)\quad & D(\alpha).q_\lambda(\alpha)\\ &= e_\lambda(\alpha)q_\lambda(\alpha) + 2\alpha\sum_{i<j}\sum_{k\geq 1}(\lambda_i-\lambda_j+2k)q_{\lambda_1}\cdots q_{\lambda_i+k}\cdots q_{\lambda_j-k}\cdots .\end{aligned}$$

Proof. Write $D(\alpha) = A(\alpha) + B(\alpha)$ with $B(\alpha) = \sum_{i,j\geq 1} h_i h_j h_{-(i+j)}$. Then $A(\alpha)$ is a derivation on V. For $q_\lambda = q_{\lambda_1}q_{\lambda_2}\cdots q_{\lambda_s}$, we have:

$$\begin{aligned}B(\alpha).q_\lambda &= \sum_l q_{\lambda_1}\cdots(B(\alpha).q_{\lambda_l})\cdots q_{\lambda_s}\\ &\qquad + \sum_{l\neq m}\sum_{i,j\geq 1} h_{-(i+j)}q_{\lambda_1}\cdots(h_i.q_{\lambda_l})\cdots(h_j.q_{\lambda_m})\cdots q_{\lambda_s}.\end{aligned}$$

Combining the action of $A(\alpha)$ and $B(\alpha)$ we have

$$\begin{aligned}D(\alpha).q_\lambda &= \sum_l q_{\lambda_1}\cdots(D(\alpha).q_{\lambda_l})\cdots q_{\lambda_s}\\ &\qquad + 2\sum_{l<m}\sum_{i,j\geq 1} h_{-(i+j)}q_{\lambda_1}\cdots q_{\lambda_l-i}\cdots q_{\lambda_m-j}\cdots q_{\lambda_s}\end{aligned}$$

Applying Lemma (3.8) to it finishes the proof. □

Proposition 4.2. *There exists a unique family of rational functions $\{C_{\lambda\mu}(\alpha)|\mu \geq \lambda\}$ such that*

(1) $C_{\lambda\lambda}(\alpha) = 1$ for all $\lambda \in \mathcal{P}$,

(2) $\sum_{\mu\geq\lambda} C_{\lambda\mu}(\alpha)q_\mu(\alpha)$, denoted as $Q'_\lambda(\alpha)$, is an eigenvector for $D(\alpha)$ for each $\lambda \in \mathcal{P}$.

Proof. For each partition λ, we use induction on the dominance ordering $\geq$ to prove that there is a unique family $\{C_{\lambda\mu}(\alpha)|\mu \geq \lambda\}$, such that the two properties are satisfied. First note that by setting $C_{\lambda\lambda}(\alpha) = 1$ we see the coefficient of $q_\lambda(\alpha)$ in the following expression is zero:

$$e_\lambda(\alpha)\sum_{\xi\geq\lambda} C_{\lambda\xi}(\alpha)q_\xi(\alpha) - \sum_{\xi\geq\lambda} C_{\lambda\xi}(\alpha)D(\alpha).q_\xi(\alpha). \tag{4.2}$$

This is due to the fact that the coefficient of q_λ in $D(\alpha).q_\lambda$ is $e_\lambda(\alpha)$. Now assume that for each ν such that $\lambda \leq \nu < \mu$, $C_{\lambda\nu}(\alpha)$ is already found such that the coefficients of $q_\nu(\alpha)$ in equation (4.2) are zero. We will see that $C_{\lambda\mu}(\alpha)$ is uniquely determined such that the coefficient of $q_\mu(\alpha)$ in equation (4.2) is also zero. This means that we must have the following equation

$$e_\lambda(\alpha)C_{\lambda\mu}(\alpha) - \sum_{\lambda\leq\nu\leq\mu} C_{\lambda\nu}(\alpha)R_{\nu\mu}(\alpha) = 0,$$

where $R_{\nu\omega}(\alpha)$ is defined by $D(\alpha).q_\nu(\alpha) = \sum_{\omega\geq\nu} R_{\nu\omega}(\alpha)q_\omega$ (thus we have $R_{\mu\mu}(\alpha) = e_\mu(\alpha)$). By the inductive hypothesis, $C_{\lambda\nu}(\alpha)$ have already been found except $\nu = \mu$. Then we solve the equation to determine $C_{\lambda\mu}(\alpha)$:

$$C_{\lambda\mu}(\alpha) = \frac{\sum_{\nu<\mu,|\nu|=|\lambda|} C_{\lambda\nu}(\alpha)R_{\nu\mu}(\alpha)}{e_\lambda(\alpha) - e_\mu(\alpha)}.$$

Here we note that $e_\lambda(\alpha) - e_\mu(\alpha) \neq 0$ as $\lambda < \mu$ by the following well-known Lemma 4.3. This finishes the existence and uniqueness of the family of $C_{\lambda\mu}(\alpha)$'s. □

Lemma 4.3. *For two partitions λ, μ with $\lambda > \mu$, we have $e_\lambda(\alpha) \neq e_\mu(\alpha)$.*

Now it is clear that Theorem 3.2 follows by combining Lemma 3.4, Proposition 4.1 and Proposition 4.2.

We can rewrite the action of the differential operator $D(\alpha)$ on the basis of q_λ. Recall that a raising operator is a product of simple raising operator R_{ij} on partitions:

$$R_{ij}\lambda = (\lambda_1, \cdots, \lambda_i + 1, \cdots, \lambda_j - 1, \cdots, \lambda_l) \tag{4.3}$$

We also define the action of a raising operator on q_λ by its action on the associated partition λ, i.e., $R_{ij}q_\lambda = q_{R_{ij}\lambda}$.

Using the raising operator on the basis $\{q_\lambda\}$, we can rewrite the action easily as follows.

Corollary 4.4. *The action of $D(\alpha)$ on the basis q_λ is given by*

$$\begin{aligned} D(\alpha)q_\lambda &= G_\lambda(R_{ij})q_\lambda \\ &= \left[e_\lambda(\alpha) + 2\alpha \sum_{1\leq i<j\leq l(\lambda)} \left(\frac{(\lambda_i - \lambda_j)R_{ij}}{1 - R_{ij}} + \frac{2R_{ij}}{(1 - R_{ij})^2}\right)\right] q_\lambda. \end{aligned}$$

Proof. Note that

$$R_{ij}^k q_\lambda = q_{\lambda_1} \cdots q_{\lambda_i+k} \cdots q_{\lambda_j-k} \cdots q_{\lambda_l},$$

and $R_{ij}^k q_\lambda = 0$ whenever $k > min(\lambda_i, \lambda_j)$. It follows that

$$\begin{aligned} &D(\alpha)q_\lambda \\ &= e_\lambda(\alpha)q_\lambda + 2\alpha \sum_{i<j}\sum_{k\geq 1}(\lambda_i - \lambda_j + 2k)R_{ij}^k q_\lambda \\ &= e_\lambda(\alpha)q_\lambda + 2\alpha \sum_{i<j}\left((\lambda_i - \lambda_j)\frac{R_{ij}}{1 - R_{ij}} + \frac{2R_{ij}}{(1 - R_{ij})^2}\right) q_\lambda, \end{aligned}$$

where we used $\sum_{k\geq 1} kR^k = \frac{R}{(1-R)^2}$. □

When λ is a rectangle, we have

$$D(\alpha)q_\lambda = \left[e_\lambda(\alpha) + \binom{l(\alpha)}{2}\frac{4R_{ij}}{(1 - R_{ij})^2}\right] q_\lambda, \tag{4.4}$$

where $i < j$ is a fixed pair. The formula clearly shows that the case of rectangular shapes are special.

Remark 4.5. We will derive a raising-operator-like formula (see corollary 5.16) for the Jack symmetric functions at the end of next section.

5. Applications of the differential operator $D(\alpha)$

Note that $D(\alpha)$ is self-adjoint and the bases $q_\lambda(\alpha)$'s and m_λ's are dual, most properties involving with $D(\alpha)$ and $q_\lambda(\alpha)$'s can be passed to those about $D(\alpha)$ and m_λ's. We list some of these properties with proofs omitted.

Proposition 5.1. *There exists a unique family of rational functions $\{B_{\lambda\mu}(\alpha)|\mu \geq \lambda\}$ of α such that*

(1) $B_{\lambda\lambda}(\alpha) = 1$ for all $\lambda \in \mathcal{P}$,

(2) $\sum_{\mu\leq\lambda} B_{\lambda\mu}(\alpha)m_\mu$, denoted as $P'_\lambda(\alpha)$, is an eigenvector for $D(\alpha)$ for all $\lambda \in \mathcal{P}$.

Theorem 5.2. *For $\lambda = (\lambda_1, \lambda_2, \cdots) \in \mathcal{P}$, $P'_\lambda(\alpha) = P_\lambda(\alpha)$ if and only if the following properties are satisfied:*

(1) There are rational functions $D_{\lambda\mu}(\alpha)$ of α with $D_{\lambda\lambda}(\alpha) = 1$ such that

$$P'_\lambda(\alpha) = \sum_{\mu\leq\lambda} D_{\lambda\mu}(\alpha)m_\mu,$$

(2) $P'_\lambda(\alpha)$ is an eigenvector for $D(\alpha)$.

We know that the symmetric functions constructed in Proposition 4.2 and 5.1 are $Q_\lambda(\alpha)$ and $P_\lambda(\alpha)$ respectively, thus they are dual to each other. We can also prove this directly from their constructions.

Proposition 5.3. *We have $\langle P'_\lambda(\alpha), Q'_\mu(\alpha)\rangle = \delta_{\lambda,\mu}$.*

Proof. If $\lambda = \mu$, $\langle P'_\lambda(\alpha), Q'_\mu(\alpha)\rangle = 1$ for that $\langle m_\lambda, q_\mu(\alpha)\rangle = \delta_{\lambda,\mu}$. If $\lambda \neq \mu$, consider two cases. In the first case that λ and μ are comparable, we have $e_\lambda(\alpha) \neq e_\mu(\alpha)$, and thus

$$\begin{aligned}
&e_\lambda(\alpha)\langle P'_\lambda(\alpha), Q'_\mu(\alpha)\rangle = \langle e_\lambda(\alpha)P'_\lambda(\alpha), Q'_\mu(\alpha)\rangle \\
&= \langle D(\alpha).P'_\lambda(\alpha), Q'_\mu(\alpha)\rangle = \langle P'_\lambda(\alpha), D(\alpha).Q'_\mu(\alpha)\rangle \\
&= e_\mu(\alpha)\langle P'_\lambda(\alpha), Q'_\mu(\alpha)\rangle.
\end{aligned}$$

Hence $\langle P'_\lambda(\alpha), Q'_\mu(\alpha)\rangle = 0$ is immediate.

In the second case that λ and μ are incomparable, we do not have a ν such that $\nu \leq \lambda$ and $\nu \geq \mu$. Because that will lead to $\lambda \geq \mu$ which is a contradiction. Thus every term in $P'_\lambda(\alpha)$ is orthogonal to the terms in $Q'_\mu(\alpha)$, and $\langle P'_\lambda(\alpha), Q'_\mu(\alpha)\rangle = 0$. □

5.1. On the action of the Virasoro algebra. The Virasoro algebra is the infinite dimensional Lie algebra generated by L_n and the central element c subject to the relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-1}{12}\delta_{m,-n}c, \tag{5.1}$$

$$[L_n, c] = 0. \tag{5.2}$$

The Feigin-Fuchs realization of Virasoro algebra on Λ can be formulated as follows.

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_{n-m}a_m : -\alpha_0(n+1)a_n,$$

$$c = 1 - 12\alpha_0^2 = 13 - 6(\alpha + \frac{1}{\alpha}),$$

where the Heisenberg generators a_n are given by

$$a_{-m} = (-1)^{m-1}\frac{1}{\sqrt{2\alpha}}h_{-m}, \qquad m > 0,$$

$$a_m = (-1)^{m-1}\sqrt{\frac{2}{\alpha}}h_m, \qquad m > 0,$$

$$a_0 = \alpha' Id,$$

$$\alpha_0 = (\frac{\sqrt{2\alpha}}{2} - \frac{1}{\sqrt{2\alpha}}),$$

where we assume $\alpha > 0$. The new operators satisfy the standard relations:

$$[a_m, a_n] = m\delta_{m,-n}I. \tag{5.3}$$

For $n > 0$, we have $L_{n+2} = (-1)^n(n!)^{-1}(adL_1)^nL_2$ by (5.1), thus to know the action of L_n we only need to know that of L_1 and L_2, which is explicitly given as follows:

$$L_1 = -\alpha^{-1}M_1 + \big(\alpha'\sqrt{\frac{2}{\alpha}} - 2(1-\frac{1}{\alpha})\big)h_1, \tag{5.4}$$

$$L_2 = \alpha^{-1}M_2 + (3\alpha_0 - \alpha')\sqrt{\frac{2}{\alpha}}h_2 + \alpha^{-1}h_1h_1, \tag{5.5}$$

where for non-negative integer n, we define

$$M_{-n} = \sum_{i\geq 1} h_{-i-n}h_i,$$

$$M_n = \sum_{i\geq 1} h_{-i}h_{i+n},$$

thus $M_{-n}^* = M_n$ for $n \in \mathbb{Z}$.

Note that we have $h_2 = \alpha^{-1}(J_2^* - h_1^2)$ and $h_1 = J_1^*$, where J_n^* is the conjugate of J_n, which can be taken as a multiplication operator. The

action of J_n on Jack functions, known as Pieri formula, was discovered by Stanley. Also note that L_0 is a scalar multiplier on each $\Lambda_F(m)$, $L_{-(n+2)} = n!)^{-1}(adL_{-1})^n.L_{-2}$, and $h^*_{-n} = h_n$, to know the action of Virasoro algebra on Jack functions, we only need to know those of M_1 and M_2. This is given by the following proposition.

Proposition 5.4. *For any pair of partitions μ, λ, we have*

$$\langle M_1.J_\lambda, J_\mu\rangle = (2\alpha)^{-1}(e_\lambda(\alpha) - e_\mu(\alpha) - e_{(1)}(\alpha))\langle J_1^*.J_\lambda, J_\mu\rangle, \tag{5.6}$$

$$\begin{aligned}\langle M_2.J_\lambda, J_\mu\rangle = (4\alpha^2)^{-1}(e_\lambda(\alpha) - e_\mu(\alpha) - e_{(2)}(\alpha))\langle J_2^*.J_\lambda, J_\mu\rangle \\ - \alpha^{-1}\langle M_1 J_1^*.J_\lambda, J_\mu\rangle.\end{aligned} \tag{5.7}$$

Proof. As in the proof of Proposition 4.1, we have

$$\begin{aligned}D(\alpha).(J_\mu J_\nu) = (D(\alpha).J_\mu)J_\nu + J_\mu(D(\alpha).J_\nu) \\ + 2\sum_{j\geq 1}(M_{-j}.J_\mu)(h_j.J_\nu).\end{aligned}$$

Notice that $h_n.J_m(\alpha) = \frac{m!}{(m-n)!}\alpha^n J_{m-n}(\alpha)$, and that $D(\alpha).J_\lambda = e_\lambda(\alpha)J_\lambda$ for any partition λ. Combining these into the case of $n = 1, 2$ in the following equation finishes the proof:

$$\langle D(\alpha).(J_\mu J_n), J_\lambda\rangle = \langle J_\mu J_n, D(\alpha).J_\lambda\rangle.$$

$\square$

Remark 5.5. In [**SAFR**], the action of L_n on Jack function was conjectured in an iterative formular. Our action of M_1 and M_2 given in Proposition 5.5 partly confirms their formula.

It is known [**FF**] that the subspace spanned by the singular vectors of fixed degree is of dimension one, thus the following recovers the result of [**MY**].

Corollary 5.6. *For a partition λ, J_λ is a singular vector of the representation of the Virasoro algebra if and only if $\lambda = (r^s)$ and $\alpha' = (r+1)\sqrt{2\alpha}/2 - (1+s)/\sqrt{2\alpha}$ for some pair of positive integers (r, s).*

Proof. To prove the necessity, we have

$$\langle L_1.J_\lambda, J_\mu\rangle = \phi(\lambda, \mu, \alpha')\langle J_1^*.J_\lambda, J_\mu\rangle,$$

where $\phi(\lambda, \mu, \alpha') = -(2\alpha^2)^{-1}(e_\lambda(\alpha) - e_\mu(\alpha) - e_{(1)}(\alpha)) + (\alpha'\sqrt{2/\alpha} - 2(1 - 1/\alpha))$. Note that the Jack functions in the expansion of $J_1^*.J_\lambda$ are labeled by partitions coming from λ by removing one of the square. If λ is not of rectangular shape, there would be at least two such partitions, say μ^1 and μ^2 with $\mu^2 < \mu^1$. Thus $\phi(\lambda, \mu^1, \alpha') \neq \phi(\lambda, \mu^2, \alpha')$, and at least one of $\langle L_1.J_\lambda, J_\mu^1\rangle$ or $\langle L_1.J_\lambda, J_\mu^2\rangle$ is non-zero, which means J_λ is not a singular vector. This proves that if J_λ is a singular vector, $\lambda = (r^s)$ for a pair of positive integers (r, s). We then compute the action of L_1 on $J_{(r^s)}$ using 5.4, 5.6, and Pieri formula [**S**] for the action of J_1^*. After some computation,

one can see that the unique value of α' that makes $L_1.J_{(r^s)}$ vanished is $(r+1)\sqrt{2\alpha}/2-(1+s)/\sqrt{2\alpha}$.

To prove the sufficiency, we only need to verify that both the actions of L_1 and L_2 on $J_{(r^s)}$ lead to zero. The computation about L_1 is already done above, while the action of L_2 invokes 5.5, 5.6, 5.7, and Pieri formula for the action of J_1^* and J_2^*. We can show that $L_2.J_{(r^s)}$ also vanishes. □

5.2. Determinant formulae for Jack symmetric functions. We use the proof of Theorem 3.2 to find new determinant formulae for Jack symmetric functions. First, note that the operator $\sum_{k\geq 1} h_{-k}h_k$ is a scalar multiplier on each $\Lambda(m)$, for simplicity we can replace the operator $D(\alpha)$ with $D'(\alpha)=(2\alpha)^{-1}D(\alpha)-\sum_{k\geq 1} h_{-k}h_k$, we then have

$$D'(\alpha).Q_\lambda(\alpha)=e'_\lambda(\alpha)Q_\lambda(\alpha),$$

where

$$e'_\lambda(\alpha)=\frac{1}{2}\alpha\sum_i \lambda_i^2-\sum_i i\lambda_i \tag{5.8}$$

for $\lambda=(\lambda_1,\lambda_2,\cdots)$. Note that $\lambda>\mu$ implies that $e'_\lambda(\alpha)-e'_\mu(\alpha)$ is of the form $a\alpha+b$, with $a,b\in\mathbb{Z}_{>0}$. Next, we notice that the action of $D(\alpha)$ or $D'(\alpha)$ on $q_\lambda(\alpha)$ can be refined. For this purpose we have the following.

Definition 5.7. Let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_s)$ be a partition, assume that $i<j$ and $\lambda_j\geq k>0$, we define the action of $r_j^i(k)$ on λ as moving k squares from the jth row to the ith row, then rearranging the rows in decreasing order to get a new partition, i.e. $r_j^i(k).\lambda$ is the rearrangement of $(\lambda_1,\cdots,\lambda_i+k,\cdots,\lambda_j-k,\cdots,\lambda_s)$ in decreasing order. We call $r_j^i(k)$ a *moving up operator* for λ, and define the moving up of λ as the set

$$M^*(\lambda)=\{r_j^i(k).\lambda | r_j^i(k) \text{ is a moving up operator of } \lambda\}.$$

We also define the moving down of μ as the set $M_*(\mu)=\{\lambda|\mu\in M^*(\lambda)\}$.

For later use, we would like to give a refinement of Proposition 4.1 using the deformation $D'(\alpha)$ of $D(\alpha)$.

Lemma 5.8. *Let $\lambda=(\lambda_1,\lambda_2,\cdots)$, we have*

$$D'(\alpha).q_\lambda(\alpha)=\sum_\mu r_{\lambda\mu}q_\mu(\alpha),$$

$$D'(\alpha).m_\mu=\sum_\lambda r_{\lambda\mu}m_\lambda,$$

where for $\mu=r_j^i(k).\lambda$,

$$r_{\lambda\mu}=(1+\delta_{\lambda_i\lambda_j})^{-1}m_{\lambda_i}(\lambda)(m_{\lambda_j}(\lambda)-\delta_{\lambda_i\lambda_j})(\lambda_i-\lambda_j+2k),$$

$r_{\nu\nu}=e'_\nu(\alpha)$ and $r_{\lambda\mu}=0$ otherwise.

PROOF. The first equality is coming from Proposition 4.1. Noting that $D'(\alpha)$ is the self-adjoint and that m_λ's is dual to q_λ's, we have the second equality. □

In [**LLM**], Jack function was given in a determinant of a matrix with entries being monomials, and equivalently a recursion formula is given. In the following we have a similar formula expressing Jack function as a determinant in terms of q_λ's. We remark that we can easily find the formula in terms of monomial symmetric functions as well in a different way.

For a partition λ, let $\lambda = \mu^1 <^L \mu^2 <^L \cdots <^L \mu^s$ be all the partitions greater than λ, arranged in lexicographic order. Set matrix $M_\lambda = (r_{ij})_{s\times s}$, where $r_{ij} = r_{\mu^j\mu^i}$ as defined in Lemma 5.8, we have

Theorem 5.9. *Set $Q_\lambda(\alpha) = \sum_i C_{\lambda\mu^i}(\alpha)q_{\mu^i}(\alpha)$, then the vector*

$$X_\lambda = (C_{\lambda\mu^1}(\alpha), C_{\lambda\mu^2}(\alpha), \cdots, C_{\lambda\mu^s}(\alpha))^t$$

satisfies

$$M_\lambda X_\lambda = e'_\lambda(\alpha)X_\lambda.$$

And we have the determinant formula for Jack functions:

$$Q_\lambda(\alpha) = c_\lambda det N_\lambda,$$

where $c_\lambda = \prod_{i=2}^s (e'_{\mu^i}(\alpha) - e'_{\mu^1}(\alpha))^{-1}$, and N_λ is the matrix $M_\lambda - e'_\lambda(\alpha)Id$ with the first row replaced by $(q_{\mu^1}(\alpha), q_{\mu^2}(\alpha), \cdots, q_{\mu^s}(\alpha))$.

PROOF. Evaluating the coefficient of $q_{\mu^k}(\alpha)$ in

$$e'_\lambda(\alpha)Q_\lambda(\alpha) = \sum_i C_{\lambda\mu^i}(\alpha)D'(\alpha).q_{\mu^i}(\alpha),$$

by Lemma 5.8, we have

$$\sum_i C_{\lambda\mu^i} r_{\mu^i\mu^k} = e'_\lambda(\alpha)C_{\lambda\mu^k}.$$

This is the kth row of $M_\lambda X_\lambda = e'_\lambda(\alpha)X_\lambda$. Thus X_λ is a solution to the system of linear equations $(M_\lambda - e'_\lambda(\alpha)Id)X = 0$, note that the coefficient matrix $A = (M_\lambda - e'_\lambda(\alpha)Id)$ has co-rank 1 and its first row is zero, an elementary result of linear algebra says that the solutions are proportional to $(A_{11}, A_{12}, \cdots, A_{1s})^t$, where A_{ij} is the algebraic cofactor of a_{ij} in $A = (a_{ij})_{s\times s}$. Note also that $A_{1k} = (N_\lambda)_{1k}$, and consider the coefficient of q_λ proves the second equation. □

5.3. Generalized rising operator formula for Jack functions. An equivalent form of the determinant formula is the following iterative formula for the coefficients of Jack functions in terms of generalized homogeneous functions. This formula is sometimes more convenient to use.

Theorem 5.10. *Let $Q_\lambda(\alpha)=\sum_{\mu\geq\lambda}C_{\lambda\mu}(\alpha)q_\mu(\alpha)$, for $\mu>\lambda$ we have*

$$C_{\lambda\mu}(\alpha)=\frac{\sum_{\lambda\leq\nu\in M_*(\mu)}C_{\lambda\nu}(\alpha)r_{\nu\mu}}{e'_\lambda(\alpha)-e'_\mu(\alpha)}.$$

As an application of Theorem 5.10, we have the following explicit formula for two-row or two-column Jack functions.

Proposition 5.11. [**JJ, S**] *For $\lambda^0=(r,s)$, with $r\geq s$, $a=r-s$, set $\lambda^i=(r+i,s-i)$ for $0\leq i\leq s$. We have*

$$Q_{\lambda^0}(\alpha)=\sum_{s\geq i\geq 0}a_i(\alpha)q_{\lambda^i}(\alpha), \tag{5.9}$$

$$J_{(\lambda^0)'}(\alpha)=\sum_{s\geq i\geq 0}b_{s-i}(\alpha)m_{(\lambda^i)'}, \tag{5.10}$$

where $a_0(\alpha)=1$ and for $i\geq 1$,

$$a_i(\alpha)=(-1)^i(a+2i)\frac{(a+1)\cdots(a+i-1)}{i!}\frac{(1-\alpha)\cdots(1-(i-1)\alpha)}{(1+(a+1)\alpha)\cdots(1+(a+i)\alpha)},$$

$$b_k(\alpha)=(s+r-2k)!\prod_{1\leq j\leq k}(s+1-j)(\alpha+j).$$

Proof. For the statement about $Q_{\lambda^0}(\alpha)$, we have $e'_{\lambda^0}(\alpha)-e'_{\lambda^i}(\alpha)=-i(1+(a+i)\alpha)$. Let's consider $\lambda=\lambda^0$ in Theorem 5.10, we find

$$a_i(\alpha)=\frac{a+2i}{-i(1+(a+i)\alpha)}\sum_{j=0}^{i-1}a_j(\alpha),$$

where we use the fact that $r_{\lambda^j\lambda^i}=a+2i$ for $j<i$.

To finish the proof, we prove that the assignment of $a_i(\alpha)$ in the statement satisfies the following equality:

$$\sum_{j=0}^{i-1}a_j(\alpha)=a_i(\alpha)\frac{-i(1+(a+i)\alpha)}{a+2i}. \tag{5.11}$$

In fact, for $i=1$, it is immediate. If (5.11) is true, then we have

$$\begin{aligned}\sum_{j=0}^{i}a_j(\alpha)&=a_i(\alpha)\frac{-i(1+(a+i)\alpha)}{a+2i}+a_i(\alpha)\\&=a_i(\alpha)\Big(\frac{-i(1+(a+i)\alpha)}{a+2i}+1\Big)\\&=a_i(\alpha)\frac{(a+i)(1-i\alpha)}{a+2i}\\&=a_{i+1}(\alpha)\frac{-(i+1)(1+(a+i+1)\alpha)}{a+2(i+1)}.\end{aligned}$$

For the statement about $J_{(\lambda^0)'}$, the proof is exactly the same and is omitted. $\square$

In [**S**], Stanley conjectured that the Littlewood-Richardson coefficient $C^{\lambda}_{\mu\nu}(\alpha) = \langle J_\mu(\alpha)J_\nu(\alpha), J_\lambda(\alpha)\rangle$ is a polynomial of α with nonnegative integer coefficients. Except a few special case (for example μ is a one row partition), this conjecture is believed to be open. In the following, we will give a combinatorial formula for the coefficients.

First, we will give a combinatorial formula for the Jack symmetric functions based on the iteration formula in Theorem 5.10. To do this, we need the following definition.

Definition 5.12. For a sequence of partitions $\delta = (\lambda^0, \lambda^1, \cdots, \lambda^s)$, we say that δ is a moving up filtration of partitions starting from λ^0 and ending at λ^s if $\lambda^i \in M^*(\lambda^{i-1})$ for $i = 1, 2, \cdots, s$. For such a filtration we denote its initial partition as $st(\delta) = \lambda^0$ and the last partition as $ed(\delta) = \lambda^s$. Assume that $\lambda \geq \lambda^s$, and $\lambda^0 \geq \mu$ we define

$$f^\lambda(\delta) = \prod_{i=0}^{s-1} \frac{r_{\lambda^i\lambda^{i+1}}}{e'_\lambda(\alpha) - e'_{\lambda^i}(\alpha)},$$

$$f_\mu(\delta) = \prod_{i=0}^{s-1} \frac{r_{\lambda^i\lambda^{i+1}}}{e'_\mu(\alpha) - e'_{\lambda^{i+1}}(\alpha)}.$$

Theorem 5.13. *Let $J_\lambda(\alpha) = \sum_\mu v_{\lambda\mu}(\alpha)m_\mu$, $Q_\mu(\alpha) = \sum_\lambda C_{\mu\lambda}(\alpha)q_\lambda(\alpha)$ for $\mu < \lambda$ we have*

$$v_{\lambda\mu}(\alpha) = v_{\lambda\lambda}(\alpha)\sum f^\lambda(\delta),$$

$$C_{\mu\lambda}(\alpha) = C_{\mu\mu}(\alpha)\sum f_\mu(\delta),$$

where both sums are over all moving up filtrations of partitions δ from μ to λ.

Note that $C_{\mu\mu} = 1$ and $v_{\lambda\lambda}(\alpha) = \sum_{s\in\lambda} h^\lambda_*(s)$. Also notice that $v_{\lambda\mu}(\alpha)$ is an integral polynomial of α by Theorem 2.4, we have:

Corollary 5.14. *The coefficient $v_{\lambda\mu}$ is the product of integral polynomials of the form $a_i\alpha + b_i$ if there is a unique moving up filtration from μ to λ.*

Note that we can also write theorem 5.13 in another form as

$$J_\lambda(\alpha) = v_{\lambda\lambda}(\alpha)\sum_{ed(\delta)=\lambda} f^\lambda(\delta)m_{st(\delta)},$$

$$Q_\mu(\alpha) = C_{\mu\mu}(\alpha)\sum_{st(\delta)=\mu} f_\mu(\delta)q_{ed(\delta)}(\alpha).$$

Using this formula, we can give a combinatorial formula for the Littlewood-Richardson coefficients of Jack functions.

Theorem 5.15. *We have*

$$\langle Q_\mu(\alpha)Q_\nu(\alpha), J_\lambda(\alpha)\rangle = v_{\lambda\lambda}(\alpha)\sum_{\delta_1,\delta_2,\delta} f_\mu(\delta_1)f_\nu(\delta_2)f^\lambda(\delta),$$

where the sum is over all triples of moving up filtrations $(\delta_1,\delta_2,\delta)$ *such that* $st(\delta_1)=\mu$, $st(\delta_2)=\nu$, $ed(\delta)=\lambda$, *and* $st(\delta)=ed(\delta_1)\cup ed(\delta_2)$.

We have the following rising-operator-like formula for Jack functions as a corollary of Theorem 5.13.

Corollary 5.16. *We have, for any partition* λ,

(5.12)

$$Q_\lambda(\alpha) = \sum_{(\underline{r})}\prod_{t=1}^{l}\frac{(r_{t-1}\cdots r_2r_1.\lambda)_{i_t}-(r_{t-1}\cdots r_2r_1.\lambda)_{j_t}+2k_t}{e'_\lambda(\alpha)-e'_{r_t\cdots r_1.\lambda}(\alpha)}q_{r_l\cdots r_2r_1.\lambda}(\alpha),$$

where the sum is over all sequences $(\underline{r})=(r_l,\cdots,r_2,r_1)$, *here* r_p *denotes the moving up operator* $r_{j_p}^{i_p}(k_p)$ *(see Definition 5.7), and* $e'_\lambda(\alpha)$ *is given in (5.8). When* $l=0$ *it corresponds to the term* $q_\lambda(\alpha)$.

Note that $q_{r_j^i(k).\lambda}=R_{ij}^k.q_\lambda$, using the usual rising operator as in Corollary 4.4 (Thus the summands corresponding to $l\le 1$ are essentially given in rising operator formula). Like the usual rising operator formula, for each $\mu\ge\lambda$, only finitely many sequences $(r_l,\cdots,r_2,r_1)$ contribute to the term q_μ. In this sense our raising operator formula generalizes the canonical Schur case to the Jack case. The difference from a usual raising operator formula is that one needs to rearrange the parts before the next action.

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